On Convergence of the Upper Bound on the Ratio of Gain to Quality Factor

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***Abstract*—An antenna's practical far-field distance can be estimated from the upper bound on the ratio of its gain to quality factor. This upper bound is an infinite series that can be truncated based on the desired accuracy. We investigate the convergence properties of this bounding series. We find that the number of terms required for convergence depends on the antenna's electrical radius in a way similar to the Wiscombe criterion used in Mie scattering theory. For typical experimental accuracy requirements, such convergence can significantly reduce the effective far-field distance.**

I. Introduction

In antenna metrology, knowledge of where the far-field behavior of an antenna begins is of paramount importance. In principle, the far-field pattern of an antenna or an antenna array is fully formed only at infinity. In practice, however, the angular field distribution of the radiated wave becomes essentially independent of the distance from the antenna at some finite effective far-field distance (EFFD). Of course, different portions of the angular field distribution exhibit different EFFD. The EFFD for the main lobe of the angular field distribution is shorter than the one for the side lobes [1], [2]. Furthermore, within a given accuracy, the EFFD for the main lobe can be approximated from the number of partial waves (modes) required to describe the radiated wave in the vector spherical harmonics basis. There are numerous approaches available to estimate the required number of modes in this basis. One interesting approach is to use the upper bound on the ratio of antenna's gain to its quality factor to give an estimated number of modes [3]. This upper bound is given by an infinite sum that can be truncated after N terms provided the remainder is sufficiently small.

In this paper, we present a systematic study of the convergence properties of the above-discussed upper bound. For a given relative error we establish the functional dependence of N on the electrical radius of the antenna or antenna array.

II. Problem Statement

The upper bound on the ratio of an antenna's gain to its quality factor is given by [4, §4.6], [5]

$$w(\rho) = \sum_{n=1}^{\infty} \frac{4(2n+1)}{u_n(\rho) + v_n(\rho)}, \tag{1a}$$

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where

$$u_n(\rho) = 2\rho - \|h_n(\rho)\|^2 \left[\rho^3 + 2(n+1)\rho\right] - \rho^3 \|h_{n+1}(\rho)\|^2 + (2n+3)\rho^2 \left[j_n(\rho) j_{n+1}(\rho) + y_n(\rho) y_{n+1}(\rho)\right] \tag{1b}$$

and

$$v_n(\rho) = 2\rho - \rho^3 \left[\|h_n(\rho)\|^2 - j_{n-1}(\rho) j_{n+1}(\rho) - y_{n-1}(\rho) y_{n+1}(\rho)\right]. \tag{1c}$$

In (1), j_n (y_n) denotes the spherical Bessel function of the first (second) kind, h_n denotes the spherical Hankel function of the first or second kind, and ρ denotes the electrical radius of the antenna. We define the electrical radius of an antenna or an antenna array as $\rho = kr$, where k denotes the wavenumber and r denotes the radius of the smallest circumscribing sphere containing the radiating portions of the antenna or antenna array. To study the convergence properties of (1a) it is convenient to define

$$w_N(\rho) = \sum_{n=1}^{N} \chi_n(\rho), \quad \text{where} \quad \chi_n(\rho) = \frac{4(2n+1)}{u_n(\rho) + v_n(\rho)}, \tag{2}$$

and analyze how well $w_N(\rho)$ approximates $w(\rho)$ for different integer values of N. It should be noted that the order of convergence of the $\{\chi_n\}$ sequence is superlinear, *i.e.*, faster than linear but slower than quadratic. To see this, substitute the asymptotic forms of the Bessel and Hankel functions for large order but fixed argument [6, §10.19(i)] into $\chi_n(\rho)$ and compute

$$\lim_{n\to\infty} \frac{|\chi_{n+1}(\rho) - L|}{|\chi_n(\rho) - L|^{\ell}}, \tag{3}$$

where $L = \lim_{n\to\infty} \chi_n(\rho) = 0$ and $\ell = 1, 2$. The order of convergence provides some insight into the asymptotic behavior of $\chi_n(\rho)$ but it does not provide a relationship between N, ρ, and the relative error,

$$\varepsilon(\rho) = \left| \frac{w(\rho) - w_N(\rho)}{w(\rho)} \right|. \tag{4}$$

We shall establish this relationship next.

III. Wiscombe-like Criterion for $w(\rho)$

In the theory of electromagnetic scattering by canonical particles, one often uses the Wiscombe criterion to truncate the infinite series of far-field quantities, such as various cross sections and (complex) scattering amplitudes [7]–[9]. Recently, in the context of the T-matrix method, a mathematically rigorous justification of the Wiscombe criterion has been provided by Ganesh et al. [10]. These far-field quantities critically depend on the asymptotic form of the scattered wave produced by the scatterer. Similarly, $w(\rho)$ critically depends on the asymptotic form of the wave radiated by the antenna. Although the physical origins of scattered and radiated waves are different, their mathematical descriptions are very similar outside the sphere circumscribing the wave source. Thus, intuitively, we expect the truncation of the infinite series in (1a) to be given by a Wiscombe-like criterion.

As a measure of accuracy, we adopt the number of significant digits to which $w_N(\rho)$ agrees with $w(\rho)$. We denote this measure by d and its relationship to the relative error is given by $d \approx -\log_{10}(\varepsilon)$ [11]. Furthermore, we choose N such that it overestimates the minimum number of summation terms required to achieve the stated accuracy by one term at the most. In other words, either $w_N(\rho)$ or $w_{N-1}(\rho)$ approximates $w(\rho)$ to the stated accuracy.

Extensive double-precision numerical computations, which are described in the Appendix, show that if we choose N to be

$$N = \left\lceil \rho + \alpha_1 \rho^{1/3} + \alpha_0 \right\rceil, \tag{5}$$

where $\lceil\cdot\rceil$ denotes the ceiling function, then $w_N(\rho)$ agrees with $w(\rho)$ to d significant digits for $10 \le \rho \le 1000$; see Table I for the numeric values of d, α_1, and α_0. The level of accuracy afforded by (5) is useful for numerical computations but beyond the accuracy level currently achievable in a laboratory. In the state-of-the-art metrology laboratory at the National Institute of Standards and Technology (NIST), a relative error of 0.01 percent to 20 percent is more appropriate [12]. For this level of accuracy, it is sufficient to restrict the linear growth of N with respect to ρ to sublinear growth. Numerically we find that $w_N(\rho)$ with

$$N = \left\lceil \rho^{\beta} + \alpha_1 \rho^{1/3} + \alpha_0 \right\rceil, \qquad 1/3 < \beta < 1, \tag{6}$$

approximates $w(\rho)$ to the stated relative accuracy for $10 \le \rho \le 1000$; see Tables II and III for the numeric values of β, α_1, α_0, and the corresponding relative error ε. Furthermore, from Tables II and III we expect β to equal $1/3$ for a large enough relative error. Indeed, this transition occurs when $\varepsilon \approx 60\%$ and the functional dependence of N on ρ becomes $N = \alpha_1 \rho^{1/3} + \alpha_0$. This functional dependence continues as the relative error increases from roughly 60% to 100%.

The restriction on ρ to be between 10 and 1000 could be relaxed if we simultaneously relax the overestimate condition on N. If we choose N such that it overestimates the minimum number of summations terms required to achieve the stated accuracy by *two* terms at the most, then (5) and (6) hold for $1 \le \rho \le 1000$. Of course, for electrically small antennas, i.e., $\rho \ll 1$, the dependence of N on ρ is not needed because $w_1(\rho)$ or $w_2(\rho)$ provide sufficient estimate of $w(\rho)$.

TABLE I
The values of d, α_1, and α_0 associated with (5) are provided.

d	α_1	α_0	d	α_1	α_0
5	2.2	1.7	8	3.7	1.3
6	2.8	1.5	9	4.1	1.2
7	3.3	1.3	10	4.5	1.2

TABLE II
The values of d, α_1, and α_0 associated with (6) are provided.

ε in %	β	α_1	α_0	ε in %	β	α_1	α_0
0.01	0.9997	1.83	1.14	0.1	0.9983	1.19	1.20
0.02	0.9995	1.64	1.19	0.2	0.9967	1.05	1.02
0.03	0.9994	1.52	1.22	0.3	0.9952	0.97	0.88
0.04	0.9992	1.44	1.25	0.4	0.9936	0.92	0.80
0.05	0.9991	1.38	1.23	0.5	0.9921	0.87	0.76
0.06	0.9990	1.32	1.25	0.6	0.9907	0.83	0.72
0.07	0.9988	1.27	1.27	0.7	0.9893	0.78	0.71
0.08	0.9986	1.25	1.22	0.8	0.9878	0.74	0.72
0.09	0.9985	1.22	1.21	0.9	0.9865	0.71	0.72

TABLE III
The values of d, α_1, and α_0 associated with (6) are provided.

ε in %	β	α_1	α_0	ε in %	β	α_1	α_0
1	0.9851	0.67	0.75	11	0.8745	−0.66	2.19
2	0.9722	0.36	1.06	12	0.8644	−0.69	2.16
3	0.9601	0.11	1.39	13	0.8543	−0.71	2.12
4	0.9486	−0.08	1.64	14	0.8443	−0.71	2.05
5	0.9375	−0.23	1.87	15	0.8343	−0.71	1.97
6	0.9266	−0.36	2.01	16	0.8243	−0.71	1.90
7	0.9160	−0.46	2.13	17	0.8144	−0.70	1.81
8	0.9054	−0.53	2.17	18	0.8045	−0.69	1.73
9	0.8951	−0.59	2.22	19	0.7946	−0.67	1.63
10	0.8848	−0.64	2.23	20	0.7847	−0.66	1.56

IV. Summary

We analyzed the convergence properties of the upper bound on the ratio of antenna's gain to its quality factor as a function of the relative error and the electrical radius of the antenna, ρ. Through a systematic numerical study, we showed that the upper bound ratio $w(\rho)$ follows Wiscombe-like dependence on ρ for relative error below approximately 10^{-5}. For larger relative errors the dominant linear term in Wiscombe-like criterion is reduced to ρ^{β} with $1/3 \le \beta < 1$.

The relative errors in typical experimental measurements tend to be well above 10^{-5} and therefore, N associated with this level of accuracy is significantly smaller than the theoretical value of N, i.e., N such that $w_N(\rho)$ equals $w(\rho)$ to double-precision. Consequently, if one defines the practical far-field distance of the antenna in terms of N, then the effective far-field distance is significantly reduced.

Appendix
Numerical Computations

To obtain the results presented in Section III we discretized ρ in steps of 0.1 from 1 to 1000. Additionally, we truncated the sum in (1a) at $n = \lceil 2\rho + 50 \rceil$ to obtain a numerically-exact estimate of $w(\rho)$. At such a large truncation integer the denominator in (1a) overflows because of the growth of $y_n(\rho)$ for $n \gg \rho$. Thus, these terms in the sum *underflow* and can be treated as numerical zeros. In other words, for large enough n, the sum can be truncated and the result should be valid to about double-precision, i.e., 15 significant digits minus a few digits for the roundoff error. In order to limit the roundoff error we used `math.fsum` method in Python's math module[1] instead of `numpy.fsum` method in the NumPy library[1], because `math.fsum` reorders the sum to minimize the roundoff error. The spherical Bessel functions of the first and second kind were computed via the SciPy library[1] methods `scipy.special.spherical_jn` and `scipy.special.spherical_yn`, respectively, and the modulus squared of the spherical Hankel functions via $\|h_n(\rho)\|^2 = j_n^2(\rho) + y_n^2(\rho)$.

The values of α_1 and α_0 in (5) were first estimated from the ordinary least squares fit of $N - \rho = \alpha_1 \rho^{1/3} + \alpha_0$ via the statsmodels library[1] method `statsmodels.formula.api.ols`. Then, the values of α_0 and α_1 were truncated to the number of the significant digits shown in Table I, and the resulting equation was manually confirmed to satisfy the stated overestimation criterion of one or two terms on the discretized set of ρ values. A similar procedure was used to obtain β, α_1, and α_0 shown in Tables II and III. However, because the dependence of N on the fit parameter β is nonlinear, we used Levenberg–Marquardt algorithm to perform the fit. In particular, we used SciPy library[1] method `scipy.optimize.curve_fit`.

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[1]Mention of this product is not an endorsement but only serves to clarify what was done in this work.

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